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Set Valued Mappings, Continuous Selections, and Metric Projections

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1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

The paper is concerned with the existence of continuous selections for set valued mappings and, in particular, for set valued mappings which are metric projections of real normed linear spaces onto finite dimensional subspaces. Those set valued mappings which are metric projections are characterized. The concepts of *derived mappings* and *stable derived mappings* of a set valued mappings between topological spaces are introduced. These concepts are investigated for mappings whose values are convex subsets of a finite dimensional real linear space, and they are used to describe recent results of other authors concerning metric projections in spaces of continuous functions. A question posed by Deutsch is answered negatively.

Let X and Y be topological spaces and let $\mathscr{P}(Y)$ denote the set of all subsets of Y. By a set valued mapping ϕ of X into Y we mean a mapping

$$\phi: X \to \mathscr{P}(Y).$$

It is necessary to admit the possibility that $\phi(x) = \varnothing$ for some $x \in X$. We will write

$$D(\phi) = \{x \in X: \phi(x) \neq \varnothing\}.$$

The "graph" of ϕ is the set

$$G(\phi) = \bigcup_{x \in X} \{x\} \times \phi(x) \subseteq X \times Y.$$

If $\psi: X \to \mathscr{P}(Y)$ is a second set valued mapping then ψ is said to be a *submapping* of ϕ, and we write $\psi \subseteq \phi$, if $\psi(x) \subseteq \phi(x)$ for all $x \in X$. Thus ψ is a submapping of ϕ if and only if $G(\psi) \subseteq G(\phi)$, and $\psi = \phi$ if and only if $G(\psi) = G(\phi)$.

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A non-empty set valued mapping

$$\phi: X \to \mathscr{P}(Y) \backslash \{\varnothing\}$$

is *lower semi-continuous* (respectively *upper semi-continuous*) if $\{x: \phi(x) \cap U \neq \varnothing\}$ (respectively $\{x: \phi(x) \subseteq U\}$) is an open subset of X whenever U is an open subset of Y.

We now define certain submappings of a set valued mapping $\phi: X \to \mathscr{P}(Y)$. The definitions are central to the discussion. We denote by

$$\phi': X \to \mathscr{P}(Y)$$

the submapping of ϕ defined by

$$\phi'(x) = \{y \in \phi(x): x \in \operatorname{int}\{x': \phi(x') \cap U \neq \varnothing\} \text{ whenever } y \in \operatorname{int} U\}$$

(here int U denotes the interior of the subset U of Y). It is easily seen that ϕ is lower semi-continuous if and only if $D(\phi) = X$ and $\phi' = \phi$. We now define $\phi^{(\alpha)}$ for each ordinal number α by $\phi^{(0)} = \phi$, $\phi^{(\alpha+1)} = (\phi^{(\alpha)})'$, and $\phi^{(\beta)}(x) = \bigcap_{\alpha < \beta} \phi^{(\alpha)}(x)$ whenever β is a limit ordinal.

If $\phi^{(\alpha+1)} = \phi^{(\alpha)}$ then $\phi^{(\beta)} = \phi^{(\alpha)}$ for all $\beta \geqslant \alpha$. This situation must occur. The transfinite sequence $(G(\phi^{(\alpha)})$: α an ordinal) of subsets of $X \times Y$ is decreasing, and strictly decreasing until it becomes constant. Therefore, if card $\alpha >$ card $X \times Y$ then $G(\phi^{(\alpha+1)}) = G(\phi^{(\alpha)})$ and so $\phi^{(\alpha+1)} = \phi^{(\alpha)}$. The eventual value of the transfinite sequence $(\phi^{(\alpha)}: \alpha$ an ordinal) will be denoted

$$\phi^*: X \to \mathscr{P}(Y).$$

The mapping ϕ' will be called *the derived mapping* of ϕ, the mappings $\phi^{(\alpha)}$ *the derived mappings* of ϕ, and ϕ^* the *stable derived mapping* of ϕ.

A continuous selection for $\phi: X \to \mathscr{P}(Y)$ is a continuous mapping $s: X \to Y$ such that $s(x) \in \phi(x)$ for all $x \in X$. Our original concern was with the existence of continuous selections for a metric projection $P = P_M: Z \to \mathscr{P}(M)$ of a real normed linear space Z onto a finite dimensional subspace M of Z. (Throughout the paper linear spaces are real.) Such a metric projection is upper semi-continuous and its values are non-empty compact convex sets. In Section 2 a characterization is given of those mappings on finite dimensional spaces which are metric projections; it yields, for example, the fact that if X is a euclidean ball, M is a finite dimensional space, and $\phi: X \to \mathscr{P}(M) \backslash \{\varnothing\}$ is upper semi-continuous and compact convex set valued then ϕ can be realized, via a homeomorphism, as the restriction of a metric projection. Thus, in general, the restriction of the discussion to metric projections is irrelevant. We are concerned mainly with set valued mappings $\phi: X \to \mathscr{P}(Y)$, where Y is a finite dimensional real linear space and ϕ is convex set valued.

The mappings ϕ', $\phi^{(\alpha)}$, and ϕ^* are considered in more detail in Section 3. The relevance of ϕ^* to the existence of continuous selections is established by the following simple self-generated extension of Michael's celebrated selection theorem. Michael's theorem appeares in [16] but there is later literature (see, for example, [17] and references therein). Theorem 1.1 is proved in Section 3.

THEOREM 1.1. *Suppose that X is a paracompact Hausdorff space and Y is a Banach space. If $\phi: X \to \mathscr{P}(Y)\backslash\{\varnothing\}$ is a non-empty set valued mapping such that $\phi(x)$ is a closed convex subset of Y for each $x \in X$ then there exists a continuous selection for ϕ if and only if $\phi^*(x) \neq \varnothing$ for all $x \in X$. If X is completely paracompact (i.e., every open subset of X is paracompact) then $U = D(\phi^*)$ is the largest open subset U of X with the property that there exists a continuous selection for $\phi \mid U$.*

The main results of the paper are those of Section 4 which is concerned with convex set valued mappings into finite dimensional real linear spaces. In particular situations one can ask for a description of ϕ^* and for information about the first ordinal α such that $\phi^{(\alpha)} = \phi^*$. The present writer conjectures that any ordinal can occur as $\min\{\alpha : \phi^{(\alpha)} = \phi^*\}$ for some X, Y, and $\phi: X \to \mathscr{P}(Y)$. However, for significant classes of set valued mappings the situation can be dramatically different. The main result of the paper is the following theorem.

THEOREM 1.2. *Suppose that n is a positive integer, X is a topological space, and $\phi: X \to \mathscr{P}(\mathbb{R}^n)$ is a set valued mapping such that $\phi(x)$ is convex for each $x \in X$. Then $\phi^{(n)} \mid \operatorname{int} D(\phi^{(n)})$ is lower semi-continuous and, consequently*

$$\phi^* = \begin{cases} \phi^{(n)} & \text{if } D(\phi^{(n)}) \text{ is open in } X, \\ \phi^{(n+1)} & \text{if } D(\phi^{(n)}) \text{ is not open in } X. \end{cases}$$

Theorem 1.2 is, in two ways, best possible even when restricted to set valued mappings which are metric projections (the next theorem describes precisely in which ways). It is proved in Section 4.

THEOREM 1.3 (a) *There exists a real normed linear space X of dimension $2n+1$ and a subspace M of X, of dimension n, such that for the metric projection $P: X \to \mathscr{P}(M)$ of X onto M*

(1) $D(P^{(n-1)}) = X$,

(2) $P^{(n)} \neq P^{(n+1)}$.

(b) *There exists a normed linear space X of dimension $2n$ and a* *subspace* M *of* X, *of dimension* n, *such that for the metric projection* $P: X \to \mathscr{P}(M)$ *of* X *onto* M

(1)′ $D(P^{(n)}) = X$,

(2)′ $P^{(n-1)} \neq P^{(n)}$.

These results provide a negative answer to a question posed by Deutsch [10, Problem 2.16]. Deutsch and Kenderov [12] have considered convex set valued mappings into one dimensional spaces; their main result follows from the case $n = 1$ of Theorem 1.2 together with Michael's selection theorem (Theorem 1.1).

The final Section 5 concerns metric projections onto finite dimensional subspaces of spaces of continuous functions. Continuous selections for these metric projections have been considered recently by several authors (Blatter and Schumaker [5], Fischer [13], Li [15]). In Section 5 mild extensions of Li's results are described in terms of the derived mappings of the metric projections. Here we state only one of the conclusions—one which contrasts strongly with the results of Theorems 1.2 and 1.3.

THEOREM 1.4. *Let* X *be a compact Hausdorff space and let* $C(X)$ *be the space of real continuous functions on* X, *equipped with the uniform norm. Let* M *be a finite dimensional subspace of* $C(X)$ *and let* $P: C(X) \to \mathscr{P}(M)$ *be the metric projection of* $C(X)$ *onto* M. *Then* $P' \mid \operatorname{int} D(P')$ *is lower semi-continuous and*

$$P^* = \begin{cases} P' & \text{if } D(P') \text{ is an open subset of } C(X) \\ P^{(2)} & \text{if } D(P') \text{ is not open.} \end{cases}$$

(*It should be remarked that it can happen that* $P^* = P$; *that is, it can happen that* P *is lower semi-continuous* [6, 4].)

2. Which Set Valued Mappings are Metric Projections?

Let X be a real normed linear space and let M be a subspace of X. Then for $x \in X$

$$d(x, M) = \inf\{ \| x - m \| : m \in M\}$$

is the *distance of* x *from* M and

$$P: X \to \mathscr{P}(M)$$

defined by

$$P(x) = \{m \in M: \|x - m\| = d(x, M)\}$$

is the *metric projection* of X onto M.

There is now an extensive literature concerned with continuity properties of metric projections and with the existence of continuous selections. The problem have been considered both in general and in particular contexts (see, for example, the survey article [10]).

Suppose that the subspace M is finite dimensional. Then P is upper semi-continuous and for each $x \in X$ the set $P(x)$ is non-empty compact and convex. For our discussion the domain of P is relevant, it seems that the upper semi-continuity is not, and the most significant property of P is that it is convex valued. First we characterize metric projections in finite dimensional real linear spaces.

THEOREM 2.1. *Let X be a finite dimensional real linear space and let M be a subspace of X. If $P: X \to \mathscr{P}(M)$ is a set valued mapping of X into M then there exists a norm on X such that P is the metric projection of X onto M relative to that norm if and only if the following conditions are satisfied:*

(i) *$P: X \to \mathscr{P}(M)$ is upper semi-continuous,*

(ii) *$P(x)$ is non-empty, compact, and convex for each $x \in X$,*

(iii) *$P(\lambda x) = \lambda P(x)$ for all $x \in X$ and $\lambda \in \mathbb{R}$,*

(iv) *$P(x + m) = P(x) + m$ for all $x \in X$ and $m \in M$.*

The necessity of conditions (i)–(iv) is well-known and easily established. The sufficiency of the conditions will be established via a variant of the theorem.

The space X is isomorphic to a euclidean space and so we can speak of euclidean balls and spheres (of centre 0) in X and in any subspace of X. If $X = L \oplus M$ and Σ is a euclidean sphere, centre 0, in L such that $L = \mathbb{R}\Sigma$ then each $x \in X$ has a unique representation

$$x = \lambda y + m, \qquad y \in \Sigma, m \in M, \lambda \geqslant 0.$$

If P is a set valued mapping satisfying conditions (iii) and (iv) then

$$P(x) = \lambda P(y) + m.$$

Thus, P is determined by $P|\Sigma$. The mapping $P|\Sigma$ is odd:

$$P(-x) = -P(x) \qquad \text{for all } x \in \Sigma.$$

Theorem 2.1 will follow from

THEOREM 2.2. *Let $X = L \oplus M$ be a finite dimensional real linear space and let Σ be a euclidean sphere, centre 0, in L such that $L = \mathbb{R}\Sigma$. If $\phi: \Sigma \to \mathscr{P}(M)\backslash\{\emptyset\}$ is an upper semi-continuous compact convex set valued mapping which is odd then there exists a norm on X such that if P is the metric projection of X onto M relative to the norm then $\phi = P \mid \Sigma$.*

Proof. Let B be a closed euclidean ball, centre 0, in X such that $\Sigma = (\text{fr } B) \cap L$ and such that $x + M$ is tangent to B for each $x \in \Sigma$. Let

$$K = \tfrac{1}{2}B \cup \bigcup_{x \in \Sigma} (x - \phi(x)).$$

The second set on the right is the "graph" of $-\phi$ and is, by the properties of ϕ, a compact subset of X. Therefore the convex hull, co K, of K is a compact convex symmetric neighbourhood of 0 in X. (That the convex hull of a compact set is compact is a consequence of Caratheodory's theorem.)

There is a norm, $\|\cdot\|$, on X such that co K is its unit ball. It will be shown that this norm satisfies the assertion of the theorem.

Let $x \in \Sigma$ and let H be the hyperplane of support to B at x. Then $x + M \subseteq H$ and $H \cap \Sigma = \{x\}$. Then K lies in one of the half-spaces determined by H and $K \cap H = x - \phi(x)$. Therefore

$$(\text{co } K) \cap (x + M) \subseteq (\text{co } K) \cap H = x - \phi(x)$$

and so

$$\inf\{\|x + m\| : m \in M\} = 1,$$

and $\|x + m\| = 1$ for $m \in M$ if and only if $m \in -\phi(x)$. Thus $d(x, M) = 1$ and $P(x) = \phi(x)$. This completes the proof.

The proof of Theorem 1.3 depends upon the construction of examples. They will first be defined on euclidean cells (homeomorphic to a subset of Σ). We now show how the construction can be "transferred" to Σ.

Let the euclidean norm on $\mathbb{R}^n$ be denoted by $|\cdot|$. Let E^n be the euclidean ball

$$E^n = \{x \in \mathbb{R}^n : |x| \leqslant 1\},$$

let Σ^{n-1} be the $(n-1)$-sphere which is the boundary of E^n, and let Σ^{n-1}_+ and Σ^{n-1}_- be the "upper" and "lower" hemispheres; that is,

$$\Sigma^{n-1}_+ = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : |x| = 1, x_n \geqslant 0\}.$$

Let $p: \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection

$$p((x_1, ..., x_n)) = (x_1, ..., x_{n-1}).$$

Then $p \mid \Sigma^{n-1}_+$ is a homeomorphism of Σ^{n-1}_+ onto E^{n-1}.

Suppose that $\phi\colon \frac{1}{2}E^{n-1} \to \mathscr{P}(M)$ is a set valued mapping (where M is still a subspace of a linear space X). Define $S\phi\colon E^{n-1} \to \mathscr{P}(M)$ by

$$(S\phi)(x) = \begin{cases} \phi(x) & \text{if } 0 \leqslant |x| \leqslant \dfrac{1}{2}, \\ 2(1-|x|)\,\phi\left(\dfrac{1-|x|}{|x|}\,x\right) & \text{if } \dfrac{1}{2} \leqslant |x| \leqslant 1. \end{cases}$$

Define $T\phi\colon \Sigma^{n-1} \to \mathscr{P}(M)$ by

$$(T\phi)(x) = \begin{cases} (S\phi)(p(x)) & \text{if } x \in \Sigma_{+}^{n-1}, \\ -(S\phi)(p(-x)) & \text{if } x \in \Sigma_{-}^{n-1}. \end{cases}$$

Then $T\phi$ is well-defined. Let h be the homeomorphism of $\frac{1}{2}E^{n-1}$ into Σ_{+}^{n-1} such that ph is the identity mapping. The properties of the transformation T are now summarised in a lemma. Its proof (which requires Lemma 3.8) is elementary and we omit the details.

LEMMA 2.3. *There exists a transformation T which associates with each set valued mapping $\phi\colon \frac{1}{2} E^{n-1} \to \mathscr{P}(M)$ an odd set valued mapping $T\phi\colon \Sigma^{n-1} \to \mathscr{P}(M)$ in such a way that*

(i) $T\phi \mid h(\frac{1}{2}E^{n-1}) = \phi p$,

(ii) *If ϕ is upper semi-continuous then so is $T\phi$,*

(iii) *If ϕ is non-empty compact convex valued then so is $T\phi$,*

(iv) $(T\phi)^{(\alpha)} = T(\phi^{(\alpha)})$ *for every ordinal α, and* $(T\phi)^* = T(\phi^*)$,

(v) *If $s\colon \frac{1}{2}E^{n-1} \to M$ is a continuous selection for ϕ then Ts is a continuous selection for $T\phi$.*

3. ELEMENTARY RESULTS CONCERNING THE DERIVED MAPPINGS

The derived mappings ϕ', $\phi^{(\alpha)}$ (α an ordinal), and ϕ^* have been defined in Section 1. The scheme of the definition can be found in Baire's classic monograph [1]. The submapping ϕ' of ϕ was introduced in [8]. Beer [2] has defined a mapping associated with ϕ, which he would denote θ_ϕ, in the case that Y is a metric space; in fact $\theta_\phi = (\phi^-)'$ and $\phi' = \theta_\phi \cap \phi$ (when the right hand sides of the equations have their natural meanings). In this section we present some routine elementary propositions, some of which require no proof, and two constructions (Lemmas 3.8 and 3.9) which are at the heart of the constructions required to justify Theorem 1.3.

Throughout this section X and Y will be topological spaces and

$\phi: X \to \mathscr{P}(Y)$ will be a set valued mapping. On occasion Y will also be a linear space and ϕ may be convex set valued. The first proposition simply emphasises remarks included in Section 1.

PROPOSITION 3.1. (i) $D(\phi') \subseteq \text{int } D(\phi)$.

(ii) *$\phi' = \phi$ if and only if $D(\phi)$ is open and $\phi \mid D(\phi)$ is lower semi-continuous.*

(iii) $(\phi^*)' = \phi^*$.

A submapping of a set valued mapping was also defined in Section 1.

PROPOSITION 3.2. (i) *If $\psi \subseteq \phi$ then $\psi' \subseteq \phi'$ and $\psi^* \subseteq \phi^*$.*

(ii) *If $\psi \subseteq \phi$ and ψ is lower semi-continuous then $\psi = \psi^* \subseteq \phi^*$.*

PROPOSITION 3.3. *If U is an open subset of X then $(\phi \mid U)' = \phi' \mid U$ and $(\phi \mid U)^* = \phi^* \mid U$.*

PROPOSITION 3.4. *If V is a subset of Y let $\phi \cap V$ be the set valued mapping defined by $(\phi \cap V)(x) = \phi(x) \cap V$. If V is an open subset of Y then $(\phi \cap V)' = \phi' \cap V$.*

The next proposition concerns properties which are inherited by the derived mappings from ϕ.

PROPOSITION 3.5. (i) *If $\phi(x)$ is closed for each $x \in X$ then $\phi'(x)$ and $\phi^*(x)$ are closed for each $x \in X$.*

(ii) *If Y is a topological linear space and $\phi(x)$ is convex for each $x \in X$ then $\phi'(x)$ and $\phi^*(x)$ are convex for each $x \in X$.*

The next proposition concerns continuous selections for ϕ; we can then give the proof of Theorem 1.1.

PROPOSITION 3.6. (i) *If $s: X \to Y$ is a continuous selection for ϕ then $s(x) \in \phi^*(x)$ for all $x \in X$.*

(ii) *If U is an open subset of X and there exists a continuous selection for $\phi \mid U$ then $U \subseteq D(\phi^*)$.*

Proof. We can regard s as a lower semi-continuous single-point-set valued submapping of ϕ. Thus (i) is a special case of Proposition 3.2(ii). Assertion (ii) follows from (i) applied to $\phi \mid U$, using Proposition 3.3.

Proof of Theorem 1.1. If U is an open subset of X and there exists a continuous selection for $\phi \mid U$ then, by Proposition 3.6.(ii), $U \subseteq D(\phi^*)$.

Under the assumptions of the theorem, for each $x \in X$ the set $\phi^*(x)$ is a closed convex subset of Y by Proposition 3.5. If $U = D(\phi^*)$ then U is open and $\phi^* | U$ is lower semi-continuous, by Propositions 3.1 and 3.3. If U is paracompact then, by Michael's selection theorem [16], there exists a continuous selection for $\phi^* | U$, and so for $\phi | U$. The theorem is proved.

The final results of this section concern operations on set valued mappings: composition with a continuous mapping, multiplication by a continuous function, and a cone construction.

PROPOSITION 3.7. *Let X, Y, and Z be topological spaces. If $\phi: X \to \mathscr{P}(Y)$ is a set valued mapping and $p: Y \to Z$ is a mapping then $p \circ \phi: X \to \mathscr{P}(Z)$ will denote the set valued mapping defined by*

$$(p \circ \phi)(x) = p(\phi(x)).$$

If $p: Y \to Z$ is continuous then $p \circ \phi' \subseteq (p \circ \phi)'$.

Proof. Suppose that $x_0 \in X$ and $y_0 \in \phi'(x_0)$. Let V be a neighbourhood of $p(y_0)$ in Z. Then $p^{-1}(V)$ is a neighbourhood of y_0 in Y and

$$\{x \in X: (p \circ \phi)(x) \cap V \neq \varnothing\} = \{x \in X: \phi(x) \cap p^{-1}(V) \neq \varnothing\}$$

which is a neighbourhood of x_0 in X. This proves the proposition.

LEMMA 3.8. *Suppose that Y is a real normed linear space, that $\phi: X \to \mathscr{P}(Y)$ is a set valued mapping such that $\bigcup \{\phi(x): x \in X\}$ is a bounded subset of Y, and that $f: X \to \mathbb{R}$ is a continuous function. Let $f\phi: X \to \mathscr{P}(Y)$ be the mapping defined by*

$$(f\phi)(x) = f(x)\phi(x) \qquad \text{for all} \quad x \in X.$$

Then

$$(f\phi)'(x) = \begin{cases} f(x)\,\phi'(x) & \text{if } f(x) \neq 0, & (1) \\ \{0\} & \text{if } f(x) = 0 \text{ and } x \in \operatorname{int} D(\phi), & (2) \\ \varnothing & \text{if } f(x) = 0 \text{ and } x \notin \operatorname{int} D(\phi). & (3) \end{cases}$$

In particular, if $D(\phi') = X$ then $(f\phi)' = f\phi'$.

Proof. The lemma will be proved first in the case in which $f(x) \neq 0$ for all $x \in X$. Suppose that this condition is satisfied, that $x_0 \in X$ and $y_0 \in \phi'(x_0)$. Let V be a neighbourhood of $f(x_0)\,y_0$ in Y. The mapping $F: X \times Y \to Y$

defined by $F(x, y) = f(x)\, y$ is continuous. So there exist neighbourhoods W of x_0 in X and U of $y_0 \in Y$ such that $W \times U \subseteq F^{-1}(V)$. Now

$$\begin{aligned}\{x \in X : f(x)\, \phi(x) \cap V \neq \varnothing\}&\\ \supseteq \{x : f(x)\, \phi(x) \cap f(x)\, U \neq \varnothing\}&\\ \cap \{x : f(x)\, U \subseteq V\}&\\ \supseteq \{x : \phi(x) \cap U \neq \varnothing\} \cap W&\end{aligned}$$

and the latter set is a neighbourhood of x_0 since $y_0 \in \phi'(x_0)$. This proves that $f(x_0)\, y_0 \in (f\phi)'(x_0)$. Thus $f\phi' \subseteq (f\phi)'$. The reverse inclusion follows because $\phi = (1/f)(f\phi)$.

Now consider the general case. The special case applied to $\phi \mid (X \backslash f^{-1}(0))$ gives (1), by Proposition 3.3. Statements (2) and (3) follow from the definition of $(f\phi)'$, the continuity of f, and the boundedness property of ϕ. The proof of the lemma is complete.

LEMMA 3.9. *Suppose that Y is a normed linear space, Y_1 is a proper closed linear subspace of Y, $y_0 \in Y \backslash Y_1$, and $\phi: X \to \mathscr{P}(Y_1)$ is a convex set valued mapping. If $\psi: X \to \mathscr{P}(Y)$ is the set valued mapping defined by*

$$\psi(x) = \operatorname{co}(\{y_0\} \cup \phi(x)) \qquad \textit{for all} \quad x \in X,$$

then

$$\psi'(x) = \operatorname{co}(\{y_0\} \cup \phi'(x)) \qquad \textit{for all} \quad x \in X.$$

Proof. The constant mapping $x \to \{y_0\}$ and ϕ are both submappings of ψ. Therefore, by Propositions 3.2(i) and 3.5(ii),

$$\operatorname{co}(\{y_0\} \cup \phi'(x)) \subseteq \psi'(x) \qquad \text{for all} \quad x \in X.$$

Let $Z = \{(1-\lambda)\, y_0 + \lambda y_1 : \lambda \in (0, 1], y_1 \in Y_1\}$. Then ϕ can be regarded as a set valued mapping into $Z \cup \{y_0\}$, and Z is an open subset of $Z \cup \{y_0\}$. Let $p: Z \to Y_1$ be the projection of Z onto Y_1 from the point y_0; that is, $p(z)$ is the intersection of the line $y_0 + \mathbb{R}(z - y_0)$ with Y_1. Then p is continuous and $p(\psi \cap Z) = \phi$. Therefore, by Propositions 3.7 and 3.4,

$$\phi' \supseteq p \circ (\psi \cap Z)' = p \circ (\psi' \cap Z).$$

It follows that, for all $x \in X$,

$$\psi'(x) \backslash \{y_0\} = \psi'(x) \cap Z \subseteq \operatorname{co}(\{y_0\} \cup \phi'(x)).$$

The proof of the lemma is complete.

4. Convex Set Valued Mappings

Theorems 1.2 and 1.3 are proved in this section and their relation to previous work is discussed.

The proof of Theorem 1.2 requires an elementary convexity lemma. The calculations will be in terms of the euclidean distance in $\mathbb{R}^n$. If $w \in \mathbb{R}^n$, $\varepsilon > 0$ then $B(w, \varepsilon)$ will denote the open ball, centre w, of radius ε. If K and K' are subsets of $\mathbb{R}^n$ then

$$\delta(K, K') = \sup\{x \in K: d(x, K')\}$$

is the *deviation* of K from K'.

Lemma 4.1. *Let $z_0, ..., z_m$ be affinely independent points in $\mathbb{R}^m$ and let $K = \text{co}\{z_0, ..., z_m\}$. If $z'_0, ..., z'_m$ are points such that $d(z'_i, z_i) < \varepsilon$ for $i = 0, ..., m$ and $w \in K$ is a point such that $d(w, \mathbb{R}^m \backslash K) \geqslant \varepsilon$ then $w \in K' = \text{co}\{z'_0, ..., z'_m\}$.*

Proof. If $\sum_{i=0}^m \theta_i = 1$, $0 \leqslant \theta_i \leqslant 1$ for $i = 0, ..., m$, then

$$d\left(\sum_{i=0}^m \theta_i z_i, \sum_{i=0}^m \theta_i z'_i\right) < \varepsilon.$$

Therefore $\delta(K, K') < \varepsilon$. Suppose that, contrary to the assertion, $w \notin K'$. Then there exists a closed half-space H such that $K' \subseteq H$, $w \notin H$. Let w' be the point such that $[w', w]$ is orthogonal to the hyperplane fr H and

$$d(w', H) = d(w, H) + \varepsilon.$$

Then $d(w', w) = \varepsilon$ and so $w' \in K$. Also $d(w', K') \geqslant d(w', H) > \varepsilon$. This contradicts the fact that $\delta(K, K') < \varepsilon$.

Proof of Theorem 1.2. It is sufficient to prove that if $D(\phi^{(n)}) = X$ then $\phi^{(n)}$ is lower semi-continuous. The theorem then follows, by Proposition 3.3, by applying this result to $\phi \,|\, \text{int}\, D(\phi^{(n)})$.

Suppose, on the contrary, that $D(\phi^{(n)}) = X$ but that $\phi^{(n+1)} \neq \phi^{(n)}$. Then ϕ has:

Property A_0. *There exist $x \in X$ and $w \in \phi^{(n)}(x)$ such that $w \notin \phi^{(n+1)}(x)$.*

We now define a property A_j, for each $j = 1, ..., n$. Property A_0 is the first, degenerate, case. A j-simplex σ in $\mathbb{R}^n$ with vertices $z_0, ..., z_j$ will be denoted

$$\sigma = \langle z_0, ..., z_j \rangle.$$

We will denote by Int σ and $\partial\sigma$ the formal interior and boundary of the simplex:

$$\text{Int}\,\sigma = \left\{ \sum_{i=0}^{j} \theta_i z_i : 0 < \theta_i < 1 \text{ for } i = 0, ..., j, \sum_{i=0}^{j} \theta_i = 1 \right\},$$

$$\partial\sigma = \bigcup_{k=0}^{j} \text{co}\{z_0, ..., \hat{z}_k, ..., z_j\}.$$

("int" is reserved for the topological interior of a subset of a space.)

DEFINITION OF PROPERTY A_j. If $j \in \{1, ..., n\}$ then ϕ will be said to have property A_j if there exist $x \in X$, a simplex $\sigma = \langle z_0, ..., z_j \rangle$ in $\mathbb{R}^n$ such that

$$z_0 \in \phi^{(n-j)}(x), \qquad \{z_1, ..., z_j\} \subseteq \phi^{(n-j+1)}(x),$$

$\varepsilon > 0$, a point $w \in \text{Int}\,\sigma$ such that

$$w \notin \phi^{(n-j+1)}(x), \qquad d(w, \partial\sigma) = 3\varepsilon,$$

and an open neighbourhood N of x such that

$$\phi^{(n-j)}(u) \cap B(z_0, 2\varepsilon) \neq \varnothing \qquad \text{for all } u \in N.$$

A contradiction will be obtained by proving:

(i) If $j \in \{0, ..., n-1\}$ and ϕ has property A_j then ϕ has property A_{j+1},

(ii) ϕ cannot have property A_n.

Proof of (i). Suppose that ϕ has property A_0 and let x, w be as in the definition of A_0. There exists $\varepsilon > 0$ and an open neigbourhood N of x in X such that

$$\phi^{(n-1)}(u) \cap B(w, \varepsilon) \neq \phi \qquad \text{for all } u \in N,$$

and

$$d(w, \phi^{(n)}(x')) \geqslant 8\varepsilon \qquad \text{for some } x' \in N.$$

For one such $x' \in N$ choose $z_0 \in \phi^{(n-1)}(x') \cap B(w, \varepsilon)$ and $z_1 \in \phi^{(n)}(x')$. Choose $w' \in [z_0, z_1]$ such that $d(w', z_0) = 3\varepsilon$. Then x', w', $\sigma = \langle z_0, z_1 \rangle$, $\varepsilon > 0$, and the neighbourhood N of x satisfy the definition of property A_1.

Now suppose that $j \in \{1, ..., n-1\}$ and that ϕ has property A_j and let x, $\sigma = \langle z_0, ..., z_j \rangle$, $\varepsilon > 0$, $w \in \text{Int}\,\sigma$, and N be as in the definition of property A_j.

Since $w \notin \phi^{(n-j+1)}(x)$ there exists an ε', $0 < \varepsilon' < \frac{1}{10}\varepsilon$, an open neighbourhood N' of x, $N' \subseteq N$, and a point $x' \in N$ such that

$$\phi^{(n-j-1)}(u) \cap B(z_0, \varepsilon') \neq \varnothing \qquad \text{for all } u \in N',$$
$$\phi^{(n-j)}(u) \cap B(z_i, \varepsilon') \neq \varnothing \qquad \text{for all } u \in N' \text{ and } i = 1, ..., j,$$

and

$$d(w, \phi^{(n-j)}(x')) \geqslant 8\varepsilon'.$$

Choose

$$z'_0 \in \phi^{(n-j-1)}(x') \cap B(z_0, \varepsilon'),$$
$$z'_i \in \phi^{(n-j)}(x') \cap B(z_i, \varepsilon') \qquad \text{for} \quad i = 1, ..., j,$$

and

$$z'_{j+1} \in \phi^{(n-j)}(x') \cap B(z_0, 2\varepsilon).$$

Let $\sigma' = \langle z'_0, ..., z'_{j+1} \rangle$. (We can use the notation $\langle \cdots \rangle$ and $\partial\sigma'$ before it is seen that the "vertices" of σ' are affinely independent.) Let P be the orthogonal projection of $\mathbb{R}^n$ onto the affine hull of $\{z_0, ..., z_j\}$. Then $P(z'_i) \in B(z_i, \varepsilon')$ for $i = 0, ..., j$. Also, $\varepsilon' < \frac{1}{10}\varepsilon < 3\varepsilon = d(w, \partial\sigma)$. Therefore, by Lemma 4.1, $w \in \text{co}\{P(z'_0), ..., P(z'_j)\}$. If $w = \sum_{i=0}^{j} \theta_i P(z'_i)$, $\sum_{i=0}^{j} \theta_i = 1$, $0 \leqslant \theta_i \leqslant 1$ for $i = 0, ..., j$, let $w'' = \sum_{i=0}^{j} \theta_i z'_i$. Then $w'' \in \text{co}\{z'_0, ..., z'_j\}$ and $d(w, w'') < 2\varepsilon'$. It follows that

$$d(w'', \phi^{(n-j)}(x')) \geqslant d(w, \phi^{(n-j)}(x')) - d(w, w'') \geqslant 6\varepsilon'$$

and

$$d(w'', \partial\langle z'_0, ..., z'_j \rangle) \geqslant d(w, \partial\sigma) - 3\varepsilon' = 3\varepsilon - 3\varepsilon'.$$

Now let Q be the orthogonal projection of $\mathbb{R}^n$ onto the affine hull of $\{z'_0, ..., z'_j\}$. Then

$$d(Q(z'_{j+1}), z'_0) \leqslant d(Q(z'_{j+1}), z_0) + d(z_0, z'_0) < 2\varepsilon + \varepsilon',$$

and so, for $i = 1, ..., j$,

$$\begin{aligned} d(w'', \text{co}\{z'_0, ..., \hat{z}'_i, ..., z'_{j+1}\} &\geqslant d(w'', \text{co}\{z'_0, ..., \hat{z}'_i, ..., z'_j, Q(z'_{j+1})\}) \\ &\geqslant d(w'', \partial\langle z'_0, ..., z'_j \rangle) - d(Q(z'_{j+1}), z'_0) \\ &> \varepsilon - 4\varepsilon' \\ &> 6\varepsilon'. \end{aligned}$$

Furthermore $\phi^{(n-j)}(x')$ is convex, by Proposition 3.5(ii), and so

$$d(w'', \text{co}\{z'_1, ..., z'_{j+1}\}) \geqslant d(w'', \phi^{(n-j)}(x')) \geqslant 6\varepsilon'.$$

It follows from the inequalities that $z'_0, ..., z'_{j+1}$ are affinely independent and that there exists a point $w' \in \text{Int}\, \sigma'$ such that $Qw' = w''$, $d(w', w'') = 3\varepsilon'$, and $d(w', \partial\sigma') = 3\varepsilon'$. Also,

$$d(w', \phi^{(n-j)}(x')) \geqslant d(w'', \phi^{(n-j)}(x')) - d(w', w'') \geqslant 3\varepsilon' > 0,$$

so $w' \notin \phi^{(n-j)}(x')$.

Finally, we note that, for all $u \in N'$,

$$\varnothing \neq \phi^{(n-j-1)}(u) \cap B(z_0, \varepsilon') \subseteq \phi^{(n-j-1)}(u) \cap B(z'_0, 2\varepsilon').$$

Thus, x', σ', ε', w', and N' satisfy the definition of property A_{j+1} and (i) is proved.

Proof of (ii). Suppose on the contrary that ϕ has property A_n. Let x, σ, ε, w, and N be as in the definition of property A_n. Let N' be an open neighbourhood of x such that $N' \subseteq N$ and

$$\phi(u) \cap B(z_i, 2\varepsilon) \neq \varnothing \qquad \text{for all } u \in N' \text{ and } i \in \{0, ..., n\}$$

(for $i = 0$ the condition is the final condition of property A_j, $j = n$). Now since $d(w, \partial\sigma) = 3\varepsilon > 2\varepsilon$ the set

$$W = \bigcap \{\text{Int}\, \langle u_0, ..., u_n \rangle : u_i \in B(z_i, 2\varepsilon) \text{ for } i = 0, ..., n\}$$

is a neighbourhood of w, by Lemma 4.1. Also $W \subseteq \phi(u)$ for all $u \in N'$, by the choice of N' and the convexity of $\phi(u)$. Therefore $w \in W \subseteq \phi^{(1)}(x)$ which is a contradiction. The proof of Theorem 1.2 is complete.

The next theorem achieves the first stage in the proof of Theorem 1.3.

THEOREM 4.2. (a) *There exists an upper semi-continuous compact convex set valued mapping* ϕ: $[-1, 1]^n \to \mathscr{P}(\mathbb{R}^n)$ *such that*

(1) $D(\phi^{(n-1)}) = [-1, 1]^n$,

(2) $\phi^{(n)} \neq \phi^{(n+1)}$.

(b) *There exists an upper semi-continuous compact convex set valued mapping* ψ: $[-1, 1]^{n-1} \to \mathscr{P}(\mathbb{R}^n)$ *such that*

(1)′ $D(\psi^{(n)}) = [-1, 1]^{n-1}$,

(2)′ $\psi^{(n-1)} \neq \psi^{(n)}$.

Proof. (a) Let $X=[-1,1]^n$ and $Y=\mathbb{R}^n$. The space X is a subspace of $\mathbb{R}^n$ and $\mathbb{R}^n$ will be regarded as a product space, $\mathbb{R}^n=\mathbb{R}^{n-k}\times\mathbb{R}^k$, as is convenient.

Let $Y_1\subseteq\cdots\subseteq Y_n=Y$ be subspaces of Y such that $\dim Y_i=i$ for $i=1,\dots,n$, and choose $y_i\in Y_i\setminus Y_{i-1}$ for $i=2,\dots,n$.

For $k=1,\dots,n$ let Z_k be the set of points in X of the form

$$\left(\frac{1}{i_1},\dots,\frac{1}{i_{n+1-k}},0,\dots,0\right)$$

where $(i_1,\dots,i_{n+1-k})\in\mathbb{N}^{n+1-k}$ and $1<i_1\leqslant\cdots\leqslant i_{n+1-k}$, and let $Z_{n+1}=\{0\}\subseteq X$. Then each Z_k is a discrete set and its derived set (i.e., set of accumulation points) is

$$Z_k'=\bigcup_{j=k+1}^{n+1}Z_j.$$

Thus $Z_k\subseteq\mathbb{R}^{n+1-k}\times\{0\}$ and $Z_{k+1}=Z_k'\setminus(\mathbb{R}^{n-k-1}\times\{0\})$.

We now construct set valued mappings $\phi_k: X\to\mathscr{P}(Y_k)$ for $k=1,\dots,n$, such that the following conditions are satisfied:

(i)$_k$ $\phi_k(x)$ is a non-empty compact convex subset of Y_k for each $x\in X$.

(ii)$_k$ ϕ_k is upper semi-continuous.

(iii)$_k$ $\phi_k^{(k-1)}(x)\neq\varnothing$ for all $x\in X$.

(iv)$_k$ $\phi_k^{(k)}(x)=\varnothing$ for all $x\in Z_k$.

(v)$_k$ $\phi_k^{(k-1)}$ is lower semi-continuous at each point of $X\setminus Z_k$, and so $\phi_k^{(k)}(x)=\phi_k^{(k-1)}(x)$ for each $x\in X\setminus Z_k$.

Let $f\colon\mathbb{R}^n\to\mathbb{R}$ be the function defined by

$$f(x)=\begin{cases}\dfrac{2}{\pi}\tan^{-1}\dfrac{x_1}{(x_2^2+\cdots+x_n^2)^{1/2}} & \text{if } x_2^2+\cdots+x_n^2\neq 0,\\[2ex] \operatorname{sgn} x_1 & \text{if } x_2^2+\cdots+x_n^2=0.\end{cases}$$

Then f is continuous except at 0 and

$$\liminf_{x\to 0}f(x)=-1,\qquad \limsup_{x\to 0}f(x)=1.$$

Let $(\alpha_i)_{i \geq 1}$ be a sequence of positive real numbers such that $\sum_{i=1}^{\infty} \alpha_i = 1$, and let $(z_i: i = 1, 2, ...)$ be an enumeration of Z_1. Then

$$g(x) = \sum_{i=1}^{\infty} \alpha_i f(x - z_i)$$

defines a function $g: \mathbb{R}^n \to [-1, 1]$ which is continuous except at the points of Z_1. For each $i = 1, 2, ...,$

$$\liminf_{x \to z_i} g(x) = -\alpha_i, \qquad \limsup_{x \to z_i} g(x) = \alpha_i.$$

Now identify Y_1 with $\mathbb{R}$ and define $\phi_1: X \to \mathscr{P}(Y_1)$ by

$$\phi_1(x) = [\liminf_{y \to x} g(y), \limsup_{y \to x} g(y)].$$

Conditions $(\mathrm{i})_k$–$(\mathrm{v})_k$, $k = 1$, are all satisfied.

Now let $f_k: X \to \mathbb{R}$, $k = 2, ..., n$, be the continuous function defined by

$$f_k(x) = d(x, \mathbb{R}^{n-k} \times \{0\})$$

(the euclidean distance of the point x from the set $\mathbb{R}^{n-k} \times \{0\}$). Define the set valued mappings $\psi_{k+1}: X \to \mathscr{P}(Y_{k+1})$ and $\phi_{k+1}: X \to \mathscr{P}(Y_{k+1})$, for $k = 1, ..., n-1$, by

$$\psi_{k+1}(x) = \begin{cases} \mathrm{co}(\{y_{k+1}\} \cup \phi_k(x)) & \text{if } \ x = (x_1, ..., x_n) \text{ and } x_{n+1-k} \geq 0, \\ \phi_k(x) & \text{otherwise,} \end{cases}$$

and

$$\phi_{k+1} = f_{k+1} \psi_{k+1}.$$

Suppose that Conditions $(\mathrm{i})_k$–$(\mathrm{v})_k$ are satisfied for some $k \in \{1, ..., n-1\}$; they must be verified for $k+1$. Conditions $(\mathrm{i})_{k+1}$ and $(\mathrm{ii})_{k+1}$ are simple consequences of Conditions $(\mathrm{i})_k$ and $(\mathrm{ii})_k$, of the definitions and of simple properties of upper semi-continuity.

Let U_k, L_k, and H_k be the sets of points $x = (x_1, ..., x_n)$ in X such that $x_{n+1-k} > 0$, $x_{n+1-k} < 0$, and $x_{n+1-k} = 0$, respectively. Then U_k, L_k are open subsets of X and H_k is their common boundary. Also $Z_k \subseteq U_k$ and $\mathbb{R}^{n-k} \times \{0\} \subseteq H_k$.

We now calculate the derived mappings of ϕ_{k+1} in terms of those of ϕ_k. It will be shown first that

$$\psi_{k+1}^{(j)}(x) = \mathrm{co}(\{y_{k+1}\} \cup \phi_k^{(j)}(x)) \quad \text{for all } \ x \in U_k \ \text{ and } \ j = 1, ..., k, \tag{3}$$

$$\psi_{k+1}^{(j)}(x) = \phi_k^{(j)}(x) \quad \text{for all } \ x \in L_k \cup H_k \ \text{ and } \ j = 1, ..., k. \tag{4}$$

The statement (3) follows from Lemma 3.9 applied to $\psi_{k+1} | U_k$. The set L_k is open and $\psi_{k+1} | L_k = \phi_k | L_k$ so (4), for $x \in L_k$, requires no proof.

It remains to prove (4) for $x \in H_k$. From the relation $\phi_k \subseteq \psi_{k+1}$ it follows, by Proposition 3.2, that $\phi_k^{(j)} \subseteq \psi_{k+1}^{(j)}$ for all $j = 1, ..., k$. Suppose that $x \in H_k$. Then $x \in L_k^-$ and $\psi_{k+1}^{(j)}(x') \subseteq Y_k$ for all $x' \in L_k$ and for $j = 0, ..., k-1$. Therefore $\psi_{k+1}^{(j)}(x) \subseteq Y_k$ for $j = 1, ..., k$, but $y_{k+1} \notin Y_k$. An elementary calculation shows that if $y \in Y_k$ and $d(y, \mathrm{co}(\{y_{k+1}\} \cup \phi_k^{(j)}(x')))$ is small then $d(y, \phi_k^{(j)}(x'))$ is small (using the fact that, by Proposition 3.5, $\phi_k^{(j)}(x')$ is convex). It now follows from what has been said, for $j = 1, ..., k$ in turn, that (4) holds for all $x \in H_k$.

Now, by $(\mathrm{v})_k$ and Lemma 3.8 applied to $\phi_{k+1} = f_{k+1}\psi_{k+1}$, it follows from (3) and (4) that

$$\phi_{k+1}^{(k)}(x) = \begin{cases} f_{k+1}(x)\,\mathrm{co}(\{y_{k+1}\} \cup \phi_k^{(k-1)}(x)) & \text{if } x \in U_k \backslash Z_k, \\ \{f_{k+1}(x)\, y_{k+1}\} & \text{if } x \in Z_k \\ f_{k+1}(x)\, \phi_k^{(k-1)}(x) & \text{if } x \in L_k \cup H_k. \end{cases}$$

Thus $(\mathrm{iii})_{k+1}$ is satisfied. It follows easily that $(\mathrm{iv})_{k+1}$ is satisfied. It also follows, using $(\mathrm{v})_k$, that $\phi_{k+1}^{(k)}$ is lower semi-continuous at each point of $X \backslash (Z_k^- \backslash Z_k)$. Now

$$X \backslash (Z_k^- \backslash Z_k) \supseteq X \backslash (Z_{k+1} \cup (\mathbb{R}^{n-1-k} \times \{0\}))$$

and $\phi_{k+1}^{(k)}$ is lower semi-continuous at each point of $\mathbb{R}^{n-1-k} \times \{0\}$ by $(\mathrm{iii})_{k+1}$ and the properties of f_{k+1}. This proves that $(\mathrm{v})_{k+1}$ is satisfied.

This proves that Conditions $(\mathrm{i})_k$–$(\mathrm{v})_k$ are satisfied for all $k = 1, ..., n$. Then $\phi = \phi_n$ satisfies Conditions (1) (by $(\mathrm{iii})_n$) and (2) (because $D(\phi_n^{(n)}) = X \backslash Z_n$, by $(\mathrm{iv})_n$ and $(\mathrm{v})_n$, and $X \backslash Z_n$ is not open). This proves part (a) of the theorem.

(b) Let Y be a subspace of $\mathbb{R}^n$, of dimension $n-1$. By part (a) of the theorem there exists an upper semi-continuous compact convex set valued mapping ϕ: $[-1, 1]^{n-1} \to \mathscr{P}(Y)$ such that $D(\phi^{(n-2)}) = [-1, 1]^{n-1}$ and $\phi^{(n-1)} \neq \phi^{(n)}$. Choose $y \in \mathbb{R}^n \backslash Y$. Define ψ: $[-1, 1]^{n-1} \to \mathscr{P}(\mathbb{R}^n)$ by

$$\psi(x) = \mathrm{co}(\{y\} \cup \phi(x)).$$

Then, by Lemma 3.9,

$$\psi^{(j)}(x) = \mathrm{co}(\{y\} \cup \phi^{(j)}(x))$$

for all $x \in [-1, 1]^{n-1}$ and all $j = 1, 2, ...$. Therefore $\psi^{(n-1)} \neq \psi^{(n)}$. However $y \in \psi^{(n)}(x)$ for all $x \in [-1, 1]^{n-1}$ and so $D(\psi^{(n)}) = [-1, 1]^{n-1}$. The proof of the theorem is complete.

Proof of Theorem 1.3. The spaces $[-1, 1]^n$ and $[-1, 1]^{n-1}$ are homeomorphic to the euclidean balls $\frac{1}{2}E^n$ and $\frac{1}{2}E^{n-1}$, respectively. Therefore Theorem 1.3 now follows from Theorem 4.2, Lemma 2.3, and Theorem 2.2.

The results of this section relate to the work of Deutsch and Kenderov [12]. They introduced the notion of *almost lower semi-continuity* (*a.l.s.c.*) of a set valued mapping $\phi: X \to \mathscr{P}(Y)$ of a topological space X into a metric space Y. The definition will not be repeated here, but if $\phi(x)$ is compact for all $x \in X$ then ϕ is a.l.s.c. if and only if $D(\phi') = X$ (see [11, Lemma 3.1]). Almost lower semi-continuity of ϕ is a necessary condition for the existence of a continuous selection for ϕ and in some circumstances it is a sufficient condition. Pelant (unpublished, see [12]) and Beer [2] have given examples of a.l.s.c. set valued mappings for which there are no continuous selections. Deutsch [10] asked whether a.l.s.c. is a sufficient condition for a metric projection of a normed linear space onto a finite dimensional subspace to possess a continuous selection. Theorem 1.3(a) (in the light of Theorem 1.2) provides a negative answer.

From Theorems 1.1 and 1.2 there follows this positive result:

THEOREM 4.3. *Suppose that X is a paracompact Hausdorff space and that Y is an n-dimensional real linear space. If $\phi: X \to \mathscr{P}(Y)$ is a set valued mapping such that $\phi(x)$ is a closed convex subset of Y for each $x \in X$ then there exists a continuous selection for ϕ if and only if $D(\phi^{(n)}) = X$.*

The case $n = 1$ of this theorem contains [12, Theorem 2.7]. (Under the conditions of the latter theorem "2-lower semi-continuity" of ϕ is equivalent to the condition that $D(\phi') = X$.)

5. METRIC PROJECTIONS IN SPACES OF CONTINUOUS FUNCTIONS

Throughout this section P will denote a metric projection $P: C(X) \to \mathscr{P}(M)$ of the space $C(X)$ of real continuous functions on a compact Hausdorff space X, equipped with the uniform norm, onto an n-dimensional subspace M of $C(X)$. We are concerned with the results of three relatively recent papers [5, 13, 15] which consider the existence of continuous selections for P. Li [15] defines a submapping P_n of P which, it turns out, provides a description of the stable derived mapping P^* of P. We must begin with what is (apart from differences of notation and expression) Li's definition.

If $f \in C(X)$ and $Q \subseteq P(f)$ then for $\theta = 1$ and $\theta = -1$ let

$$\mathrm{crit}_\theta(f, Q) = \bigcap_{q \in Q} \{x : \theta(f(x) - q(x)) = \| f - q \|\}$$

and let $\mu_f(Q)$ be the set of $q \in Q$ which are such that

$$\text{crit}_\theta(f, Q) \subseteq \text{int}\{x \colon \theta(q(x) - p(x)) \geqslant 0\} \text{ for all } p \in Q \text{ and } \theta \in \{-1, 1\}.$$

The set $\mu_f(Q)$ is, in the terminology of [15], the set of *local maximal elements* of Q (relative to f). Suppose now that Q is a closed convex subset of $P(f)$. Then it is easily seen that either $\mu_f(Q) = \varnothing$ or $\mu_f(Q)$ is a convex extremal subset, that is a *face*, of Q. A face of a closed convex subset of a finite dimensional space is necessarily closed. If $\mu_f(Q) \neq \varnothing$ then either $\mu_f(Q) = Q$ or $\dim \mu_f(Q) < \dim Q$. However, $\dim P(f) \leqslant \dim M = n$. Therefore the sequence $P(f)$, $\mu_f(P(f))$, $\mu_f^2(P(f))$, ..., is a decreasing sequence of closed convex sets and for some k, $0 \leqslant k \leqslant n$,

$$P(f) \neq \cdots \neq \mu_f^k(P(f)) = \mu_f^{k+1}(P(f)) = \cdots .$$

In particular $\mu_f^n(P(f))$ is the smallest set in the sequence. Define a submapping P_k of P by $P_k(f) = \mu_f^k(P(f))$ for $k = 0, ..., n$. We state four theorems concerning the submapping P_n.

THEOREM 5.1. *For each $f \in C(X)$ either $P_n(f)$ is empty or $P_n(f)$ is a face of the closed convex set $P(f)$.*

THEOREM 5.2. $P' \subseteq P_n$.

THEOREM 5.3. $\text{int}\, D(P_1) = \text{int}\, D(P_n)$.

THEOREM 5.4. *$P_n \mid \text{int}\, D(P_n)$ is lower semi-continuous.*

Theorem 5.1 is simply an observation already made. It was proved in [8] that $P' \subseteq P_1$ and Theorem 5.2 extends that result; the proof uses the method of [8] and depends upon an extension of [15, Theorem 1.7]. A complete account of the proof requires repetition of material from [15] and is not included here. Theorem 5.3 follows from the arguments of [15]. Theorem 5.4 in the case that $D(P_1) = C(X)$ (and so $D(P) = C(X)$) is the main result of [15]. However, the assumption that $D(P_1) = C(X)$ is unnecessary and the arguments of [15] actually yield Theorem 5.4.

The three preceding theorems can be summarised in a single theorem which contains Theorem 1.4.

THEOREM 5.5. *If M is any finite dimensional subspace of $C(X)$, of dimension n, and P is the metric projection of $C(X)$ onto M then $P' \subseteq P_n$,*

$$\text{int}\, D(P_1) = \text{int}\, D(P_n) = \text{int}\, D(P') = D(P^*),$$

and P_n, P', and P^* coincide on $D(P^*)$.

Proof. By Theorem 5.2 and the results of Section 3

$$D(P^*) \subseteq \text{int } D(P') \subseteq \text{int } D(P_n).$$

Let $U = \text{int } D(P_n)$. Then by Theorems 5.2 and 5.4,

$$P^* \mid U \subseteq P' \mid U \subseteq P_n \mid U = (P_n \mid U)^* \subseteq (P \mid U)^* = P^* \mid U.$$

Consequently $U \subseteq D(P^*)$ and the conclusions of the theorem follow.

The results summarised in Theorem 5.5 have had a long development and can be traced through numerous papers [7, 4, 14, 8, 5, 13, 15] and in unpublished work [6, 3] (the second of which has not been seen by the present writer). The results of [5, 15] are close, except that the proof of lower semi-continuity in [15] involves ideas which are not necessary under the stronger assumptions of [5]. The results of Blatter and Schumaker [5] were obtained under the assumption that there exists a continuous selection for P. However, Fischer [13] observed that it is enough for the arguments of [5] to assume that $D(P') = C(X)$. Li's results [15] were obtained under the weaker assumption that $D(P_1) = C(X)$. Li calls a subspace M of $C(X)$ with the property that $D(P_1) = C(X)$ an *LMW-subspace* in recognition of the fact that the condition defining locally maximal elements appears in the paper by Lazar, Morris, and Wulbert [14]. A detailed account of the proof of the results described in this section, and in the form given here, is included in [9] in which the results of [18] are also discussed.

We conclude with some simple observations. The first is to note the questions whether any of the sets $D(P_1)$, $D(P_n)$, and $D(P')$ are necessarily open and whether $P' = P_n$ always. The second is that the results show that the spaces $C(X)$ are approximation-theoretically special. One can easily construct examples of a finite dimensional normed linear space X, subspace M, and metric projection $P: X \to \mathscr{P}(M)$ such that $D(P^*) = X$ but $P^*(x)$ is not always a face of $P(x)$. Also, there do exist compact Hausdorff spaces X and subspaces M of $C(X)$ for which the metric projections are not lower semi-continuous but do possess continuous selections (see [10] and references therein). In the light of this, Theorems 1.2, 1.3, and 1.4 distinguish the spaces $C(X)$.

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